the integral

$$I=\int_0^a dx\, e^{xb}(a^2+x^2)^{1/2},$$

where $a\ll 1$, $b\to\infty$. Let $y=bx$, and $(a+x)^{1/2}=(2a)^{1/2}$. Then we find

$$I=\left(\frac{2a}{b^3}\right)^{1/2}\int_0^{ab} dy\, e^y(ab-y)^{1/2}=\left(\frac{a\pi}{2b^3}\right)^{1/2}e^{ab},$$

where in the last step we let $ab\to\infty$.

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Divergence and Summability of the Many-Fermion Perturbation Series*

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We investigate the convergence of the many-fermion perturbation series and show, for the case of the square-well potential, that it is a divergent series. We bound the rate of divergence and show that, by using appropriate summation procedures, it may be summed to the physically correct sum, provided the density is low enough.

1. INTRODUCTION

THERE is a widely held view[1] that the many-fermion perturbation theory as currently formulated is "sufficiently established on theoretical grounds." The purpose of this paper is to question that view. For the sake of explicitness we will consider a system of spinless fermions interacting via a square-well potential. We first establish, in the second section, that the radius of convergence for the ground-state energy of the N-body system (at fixed density) tends to zero as N tends to infinity at least as fast as $N^{-\gamma}$, where γ is any positive number less than $\frac{1}{3}$. This result implies that the perturbation series is, at best, an asymptotic one.

In the third section we consider the complete perturbation series and bound every order. We find that it diverges no faster than a geometrical series times $(n!)$, where n is the order of the term. We also give an argument based on the BCS theory of superconductivity that, in general, the series diverges at least this fast. In the final section we consider the problem of assigning a meaning to the sum of the series and show, provided the density is low enough (small compared to the jamming density for hard spheres), that it may be summed, even though divergent, to the $\lim_{N\to\infty}E_N(V)$, where $E_N(V)$ is the energy per particle for a potential of real, positive strength V in the N-body problem. We advance some arguments to support the conjecture that the methods we present give the physically correct sum in general when the physical system has no long-range order.

2. THE DIVERGENCE OF THE PERTURBATION SERIES

In this section we shall establish that the many-body perturbation series is, at best, an asymptotic series and not a convergent one, and estimate approximately the angular region in which it is asymptotic. The first important point is, that as the number of particles N tends to infinity, each order in the Rayleigh-Schrödinger perturbation series for E/N, energy per particle, tends to a finite limit. This was first asserted by Brueckner[2] and later proved by Goldstone.[3] The second important point, which we will discuss below, is that in the limit as N tends to infinity there occur branch points in the energy which move to the origin of the complex potential V plane.

The analysis of Cooper[4] for a simple model without kinetic energy may not be germane as it seems that he proves that the energy expansion has zero radius (or infinite in special cases) of convergence even for two particles in a box. This result is not appropriate to ordinary perturbation theory with a kinetic energy present.

In order to investigate the many-body problem with a square-well interaction, we shall first investigate the problem of a particle in a spherical box with a square-well potential of strength V near the origin. The potential is

$$\begin{aligned} &V, && 0<r<a,\\ &0, && a<r<a+b,\\ &+\infty, && a+b<r. \end{aligned} \tag{2.1}$$

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] See, for instance, H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963).

[2] K. A. Brueckner and C. A. Levinson, Phys. Rev. **97**, 1344 (1955). See also H. A. Bethe, *ibid.* **103**, 1353 (1956) for an extensive list of references.

[3] J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957).

[4] L. N. Cooper, Phys. Rev. **122**, 1021 (1961).

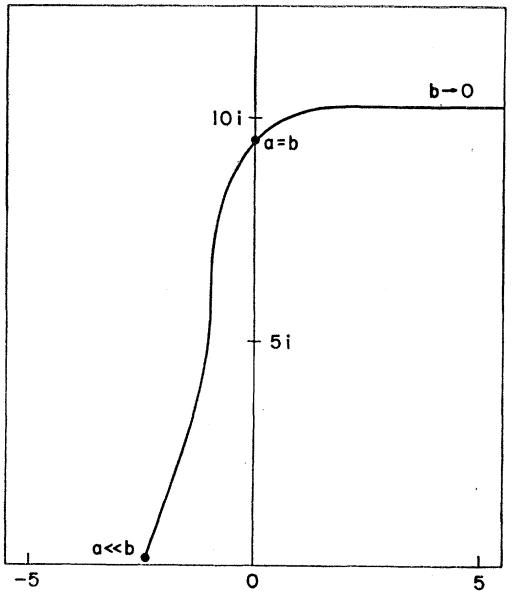

FIG. 1. Trajectory of the branch point in the ground-state energy for one particle in a box.

By solving Schrödinger's equation[5] we may easily establish that the energy of the lowest state, as a function of the potential strength, is given by the solution of

$$\tanh\left[\left(\frac{2m}{\hbar^2}(V-E)\right)^{1/2}a\right]\Big/\left[\left(\frac{2m}{\hbar^2}(V-E)\right)^{1/2}a\right]$$
$$=-(b/a)\tan\left[\left(\frac{2m}{\hbar^2}E\right)^{1/2}b\right]\Big/\left[\left(\frac{2m}{\hbar^2}E\right)^{1/2}b\right]. \quad (2.2)$$

When $V=0$, it follows readily from the trigonometric identity $\tan\alpha+\tan(\pi-\alpha)=0$ that the ground-state energy is

$$E(0)=\frac{\hbar^2}{2m}\frac{\pi^2}{(a+b)^2}. \quad (2.3)$$

As $V\to\infty$, the left-hand side of (2.2) tends to zero, the wave function is compressed into the region $a<r<a+b$, and the energy tends to

$$\lim_{V\to+\infty}E(V)=\frac{\hbar^2}{2m}\frac{\pi^2}{b^2}. \quad (2.4)$$

In fact, if $|(V-E)^{1/2}a|$ is large and we are not near the poles of the left-hand side of (2.2), then we see by approximating the right-hand side of (2.2) for $\tan x$ near $x=\pi$ by $(x-\pi)$ that $E(V)$ is approximately given by (2.4) for all V in the neighborhood of infinity, except near V real and negative. This statement is true when we start V toward infinity along the positive real axis.

[5] See, for instance, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), Sec. 15.

When $V\to-\infty$, the right-hand side of (2.2) tends to zero, the wave function falls into the region $0<r<a$, and the energy tends to

$$\lim_{V\to-\infty}[E(V)-V]=\frac{\hbar^2}{2m}\frac{\pi^2}{a^2}. \quad (2.5)$$

Again approximating $\tanh x$ near $x=i\pi$ by $(x-i\pi)$ we find (2.5) valid for large V not near V real and positive.

The difference of these two results for, say, V large and imaginary implies the existence of at least one complex conjugate pair of branch points. We may locate them as follows: It is well known that the branch points of $E(V)$ occur where $(dV/dE)=0$.[6] If we define

$$z^{1/2}=\left(\frac{2m}{\hbar^2}(V-E)\right)^{1/2}a,$$
$$t^{1/2}=\left(\frac{2m}{\hbar^2}E\right)^{1/2}b, \quad (2.6)$$

then the equation for the critical points of $E(V)$ becomes

$$\left[\frac{t^{3/2}\cos^2(t^{1/2})}{2t^{1/2}-\sin(2t^{1/2})}\right]\left[\frac{2z^{1/2}-\sinh(2z^{1/2})}{z^{3/2}\cosh^2(z^{1/2})}\right]=\left(\frac{b}{a}\right)^3, \quad (2.7)$$

subject to (2.2). If $(b/a)\gg1$, then the solution is approximately given by

$$2t^{1/2}-\sin(2t^{1/2})=0,$$
$$\cosh(z^{1/2})=0, \quad (2.8)$$

as

$$[\cos^2(t^{1/2})]/[\cosh^2(z^{1/2})]\propto(b/a)^2, \quad (b/a)\gg1. \quad (2.9)$$

Hence, in this limit we have the branch points at[6] approximately

$$V=\frac{\hbar^2}{2ma^2}\left[-\tfrac{1}{4}\pi^2+\left(\frac{a}{b}\right)^2(12.1372\pm10.3789i)\right]. \quad (2.10)$$

For the case $a=b$, $E(V)$ satisfies, by symmetry, the relation

$$E(V)-E(-V)=V, \quad (2.11)$$

and hence we expect the branch point to be on the imaginary axis. For V pure imaginary, (2.11) implies $t=-z^*$ and (2.7) reduces to

$$\mathrm{Re}\left[\frac{2z^{1/2}-\sinh(2z^{1/2})}{z^{3/2}\cosh^2(z^{1/2})}\right]=0. \quad (2.12)$$

We compute that the branch points are at the points

$$z=-6.3\pm4.6i,$$
$$V=(\hbar^2/2ma^2)(\pm9.2i). \quad (2.13)$$

[6] See, for instance, H. Kober, *Dictionary of Conformal Representations* (Dover Publications, Inc., New York, 1957), pp. xiii and 103.

As b continues to decrease until $b/a \ll 1$, we find reciprocally to (2.8) above that the branch points occur for

$$\begin{aligned} 2z^{1/2}-\sinh(2z^{1/2})&=0,\\ \cos(t^{1/2})&=0, \end{aligned} \tag{2.14}$$

which implies that, approximately,

$$z=-12.1372\pm 10.3789i, \quad t=(\tfrac{1}{2}\pi)^2,$$
$$V=\frac{\hbar^2}{2ma^2}\left[-12.1372\pm 10.3789i+(\tfrac{1}{2}\pi)^2\left(\frac{a}{b}\right)^2\right], \tag{2.15}$$

so that the branch points recede to plus infinity along a path parallel to the positive real axis. The trajectory of this branch point is shown in Fig. 1 as a function of b.

Katz[7] has illuminated the nature of these branch points. They result from the degeneracy of the ground state and the first excited state. If we cut the complex plane from these points to infinity, we have by (2.4) and (2.5) a single-valued function of V with no other branch points. However, if we join the two branch points by a cut (encircling both of them can easily be shown to leave the function unchanged), we find some of the additional branch points discussed by Katz where the ground state is degenerate with the second, third, $\cdots$ excited states.

For the higher excited states (angular momentum $=0$) the analysis is similar. For instance, for the case $b/a \gg 1$ we need the nth root for t in (2.8) for the nth excited state instead of the first root which we used for the first state. Hence, (2.10) becomes, approximately,

$$V_n \approx \frac{\hbar^2}{2ma^2}\left[-(j+\tfrac{1}{2})^2\pi^2+\left(\frac{a}{b}\right)^2\{(n-j)^2\pi^2 - \tfrac{1}{4}[\ln(4\pi(n-j))]^2 \pm i(n-j)\pi\ln(4\pi(n-j))\}\right]$$
$$j=0, 1, \cdots, n-1, \tag{2.16}$$

as long as the correction to $-(j+\frac{1}{2})^2\pi^2$ is small. There are now n pairs of branch points for the nth state. We previously found one pair for the ground state, or first state.

To return to the many-body problem, we shall show that there is at least one (pair) of branch points which tends to the origin of the V plane as N tends to infinity by traversing two paths in the complex plane to a certain point. The paths we take will never pass further from the origin than a distance which tends to zero as N tends to infinity. They will yield two different values of $E(V)$ for the same V, hence imply the existence of at least one branch point. The occurrence of a branch point stops the convergence of the Taylor series at that point. Let us consider a system defined by the Hamiltonian

$$H=\sum_{i=1}^{N}\frac{p_i^2}{2m}+\sum_{i<j}^{N} v(|r_i-r_j|), \tag{2.17}$$

which is enclosed in a large box of volume Ω, such that $\rho=N/\Omega$. The function $v(r)$ is taken to be a square well of range a and strength V.

Let us follow the energy as a function of V as V becomes progressively more attractive. We study the case of Fermi-Dirac statistics. The result here is well known.[8] The system collapses rapidly to a size of order a. The kinetic energy of the highest state ($n \propto N^{1/3}$) is proportional[8] to $N^{2/3}$. However, the potential energy of each particle is approximately NV as each particle feels the attraction of every other particle. Thus, for the collapsed state, we have for each particle a problem equivalent to one particle in a box with a square-well potential of strength $\mathcal{U}=NV$. The parameter b is related to the total volume Ω and, hence, is proportional to $(N/\rho)^{1/3}$. It becomes very large as N goes to infinity. If we go to an attractive potential of strength V proportional to $N^{-[1/(3+\epsilon)]}$, then (for N very large) the potential energy per particle [proportional to $N^{(2+\epsilon)/(3+\epsilon)}$] will dominate the kinetic energy. Also, we will be to the left of all the branch points in the complex $\mathcal{U}$ plane for the first $n=N^{1/3}$ states. The analysis equivalent to that given above for angular-momentum states different from zero is similar, and for $b/a \gg 1$ the branch points will close on the axis at the various appropriate binding energies. Returning to zero angular momentum, if we now approximate $\tanh x$ in (2.2) by $x-n_{\max}\pi i$ for x near $n_{\max}\pi i$, we may continue on $\mathcal{U}$ around the arc of a circle to the positive real axis, as long as we are careful to avoid points for which

$$\cos(t^{1/2})=0 \quad \text{or} \quad 2t^{1/2}-\sin(2t^{1/2})=0, \tag{2.18}$$

which can easily be done. At this point, the collapsed state has been continued to the positive real axis and, in terms of the energy per article, is of the order of $N^{(2+\epsilon)/(3+\epsilon)}$. As the potential energy per particle is still NV, the wave function must still be concentrated in a sphere of size a. However, if we continue from the origin straight out the positive real axis, then we know, physically, that as V tends to infinity we obtain the hard-core, Fermi gas. The energy per particle is bounded for all V, $0 \leq V < \infty$, at least for densities small compared to closest packing of hard spheres. Thus, we conclude that we must have encircled at least one branch point. As our above argument is valid for any $\epsilon > 0$, $E(V)$ must have a radius of convergence of not more than the order of $N^{-1/3}$. Hence, in the limit as N tends to infinity the radius of convergence of $E(V)$ for the potential we are considering tends to zero. As Goldstone[3] has shown formally that each term in the

[7] A. Katz, Nucl. Phys. **29**, 353 (1962).

[8] See, for example, J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), Chap. 3, Sec. 4.

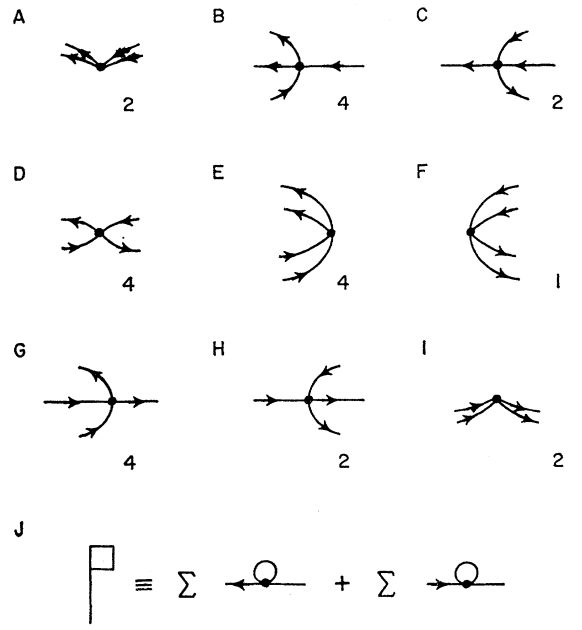

FIG. 2. Hugenholtz vertices with statistical weight factors.

expansion $E(V)/N$ is finite, we conclude that the expansion is a divergent one.

Our information on the direction in which the closest singularities approach the origin and, hence, the angle in which the series is asymptotic, is less definite. However, if we assume that, even though for $\mathcal{U}$ of the order of -1 we are nowhere near the collapsed state, the angular distribution of the correct many-body branch points is somewhat like those for the collapsed state problem, we get a physically fairly reasonable result. Looking, for low density, at the closest state of widest angle, we have ($n \propto N^{1/3}$, $j=0$) from (2.16),

$$\mathcal{U}_n \approx \frac{\hbar^2}{2ma^2}\left(-\tfrac{1}{4}\pi^2 + Aa^2\rho^{2/3} \pm iB\frac{\ln N}{N^{1/3}}\right), \qquad (2.19)$$

where A is a constant independent of N. From (2.19) it is evident that for ρ small compared to the density for the closest packing of hard spheres, that the branch points for $V=\mathcal{U}/N$ approach the axis from the negative real direction. If the second and third terms of (2.19) are not small compared to the first, then (2.19) is not valid and no conclusions can be drawn from it. Hence, at least for ρ small enough, we find that $E(V)/N$ is asymptotic in the cut plane

$$-\pi < \arg V < \pi. \qquad (2.20)$$

This does not, of course, necessarily mean $E(V)$ is analytic in the cut plane.

The case of Bose-Einstein statistics is similar to the above. However, in the collapsed state all particles occupy the lowest state so that we find the radius of convergence to be only of order $V=N^{-1}$. This result may be true for Fermi-Dirac statistics as well but we have proved only the less restrictive $V=N^{-1/3}$ instead.

We wish to point out that an argument similar to the above has been advanced previously for the case of quantum electrodynamics by Dyson.[9] He was able to show by considering e^2 negative that a phenomenon similar to the collapsed state occurs where electrons and positrons gather into separate regions of space and form a state of energy arbitrarily lower than that of the vacuum.

3. THE RATE OF DIVERGENCE OF THE PERTURBATION SERIES

In this section we shall enumerate the terms contributing to the nth order of perturbation theory and bound each term for Fermi-Dirac, spinless particles. We shall then estimate the rate of divergence of the perturbation series. It could diverge more slowly than our estimates, due to cancellation between terms, but, as we have shown in the previous section, it cannot converge. In order to enumerate all the terms it is convenient to use a diagram representation of each term. We adopt the one introduced by Hugenholtz.[10] The procedure for writing down terms of the perturbation series from diagrams is outlined by him. In the expansion for the energy, all connected diagrams with no external lines, whether or not they violate the Pauli exclusion principle,[3] are to be included. Each Hugenholtz diagram corresponds to several terms in the perturbation expansion (or several Goldstone diagrams[3] which are in one-to-one correspondence with perturbation-theory terms). The number is given by Hugenholtz as $4^n/2^m$, where n is the number of vertices and m is the number of pairs of equivalent lines. A pair of equivalent lines is two lines joining the same two vertices in the same direction. According to Hugenholtz's convention, an occupied state is represented by a line directed to the left, and a hole by a line directed to the right. The occurrence of an action by the potential is represented by a vertex. The potential function $v(k)$ introduced by Hugenholtz is, for a square well of range a and strength V,

$$v(k) = (4\pi V/k^3)[\sin(ak) - ak\cos(ak)]. \qquad (3.1)$$

In order to count all diagrams in the nth order we will give a counting rule which is equivalent, but more convenient for this purpose, to Hugenholtz's.[10] We assign a statistical weight factor to each type of vertex

TABLE I. Enumeration of diagrams.

n	1	2	3	4	5	6
F	2	4	84	4324	335 348	34 775 108
G	2	4	84	4900	454 004	60 987 716
H	0.5	0.25	1.31	16.9	328	8491

[9] F. J. Dyson, Phys. Rev. **85**, 631 (1952).
[10] N. M. Hugenholtz, Physica **23**, 481 (1957).

which correctly gives the number of Goldstone diagrams which are represented by attaching this vertex on the right to a pre-existing Goldstone diagram. In Fig. 2 we list all possible vertices, with their weight factors. Vertex J is defined as the sum of bubbles on all occupied lines plus bubbles on all hole lines. If this is not done, diagrams such as shown in Fig. 3 diverge when taken separately, but when the sum over diagrams with bubbles in all possible positions is taken, this divergence is cancelled. We will discuss this point further below.

Let us introduce a counting function $Q(n,N)$ which is equal to the number of Goldstone diagrams of order n with N external occupied-state lines (and hole lines) on the right and none on the left. The number of terms in the nth-order expansion of the energy will be related to $Q(n,0)$. If we include vertex F, we will include some disconnected diagrams. If we omit vertex F we will omit some connected ones. As we shall see, however, it is a matter of indifference to the rate of divergence whether vertex F is included (except for the first vertex) or not. We may now write recursion relations for the function Q. The subscript denotes the nature of the last vertex.

$$
\begin{aligned}
Q_A(n+1,N)&=2\binom{N}{2}Q(n,N),\\
Q_B(n+1,N)&=4\binom{N+1}{2}\binom{N+1}{1}Q(n,N+1),\\
Q_C(n+1,N)&=2\binom{N-1}{1}Q(n,N-1),\\
Q_D(n+1,N)&=4\binom{N}{1}\binom{N}{1}Q(n,N),\\
Q_E(n+1,N)&=4\binom{N+2}{2}\binom{N+2}{2}Q(n,N+2),\qquad (3.2)\\
Q_F(n+1,N)&=\max[1,Q(n,N-2)],\\
Q_G(n+1,N)&=4\binom{N+1}{1}\binom{N+1}{2}Q(n,N+1),\\
Q_H(n+1,N)&=2\binom{N-1}{1}Q(n,N-1),\\
Q_I(n+1,N)&=2\binom{N}{2}Q(n,N),\\
Q_J(n+1,N)&=(1-\delta_{N,0})Q(n,N),
\end{aligned}
$$

TABLE II. Path weights.

Type	Total weight factor	Root mean weight factor per vertex
I. $A+D+I$	$6N^2+2N+1-\delta_{N,0}$	$6(N+1/6)^2$
II. $C+H\cdots B+G$	$16N(N-1)(N+2)^2$	$4(N+3/4)^2$
III. $C+H\cdots C+H\cdots E$	$16(N-1)(N)(N+3)^2(N+4)^2$	$2.5(N+13/6)^2$
IV. $F\cdots B+G\cdots B+G$	$16(N+3)^2(N+1)^3N$	$2.5(N+3/2)^2$
V. $F\cdots E$	$(N+4)^2(N+3)^2$	$(N+7/2)^2$

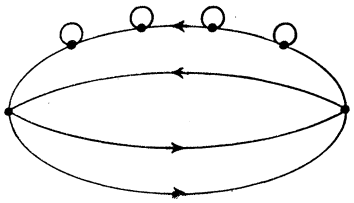

FIG. 3. A divergent bubble diagram.

where $\binom{a}{b}$ are the standard binomial coefficients, which, of course, equal the number of ways of taking a things, b at a time. Summing the terms in (3.2) we obtain the relation

$$
\begin{aligned}
Q(n+1,N)=[&(1-\delta_{N-2,0})Q(n,N-2)\\
&+4(N-1)Q(n,N-1)\\
&+(6N^2-2N+1-\delta_{N,0})Q(n,N)\\
&+4(N+1)^2NQ(n,N+1)\\
&+(N+2)^2(N+1)^2Q(n,N+2)],\quad n>1 \qquad (3.3)\\
Q(1,N)=0,\quad N\neq 2,\quad &Q(1,2)=1.
\end{aligned}
$$

The Kronecker delta $\delta_{N-2,0}$ arises from the exclusion of disconnected diagrams by not allowing vertex F to follow a state with zero lines on the right. If we drop the $Q(n,N-2)$ term, we eliminate vertex F altogether (except for the first vertex). We illustrate in Table I the number of Goldstone diagrams and a lower bound to the number of Hugenholtz diagrams. This bound is obtained by noting that no more than 4^n Goldstone diagrams may correspond to one Hugenholtz diagram. Row F is the number of Goldstone diagrams with no external lines that have only one (the first except in first order) F vertex. This number is less than or equal to the total number of connected diagrams. Row G is the number of Goldstone diagrams with no external lines which never return to the ground state. This number is greater than or equal to the total number of connected diagrams. Row H is a lower bound to the number of Hugenholtz diagrams. It is Row F over 4^n. In the first four orders it varies from about $\frac{1}{4}$ to $\frac{1}{2}$ the actual number. In order to estimate the number of diagrams for large n, we use the fact that there are the same number of lines entering on the left as leaving on the right; hence, for instance, if N is changed by $+1$ at one vertex in the diagram, it must be lowered at some subsequent vertex. In Table II we give several possible types of raising and lowering combinations, together with their weight factors and the root mean factor per vertex. The weight factor assumes N lines of each type entering from the left. The root mean weight factors are approximate. In the type designation in Table II the plus sign is used to lump all contributions of the designated types of vertex together and $\cdots$ indicates other parts of the diagram may intervene before the indicated level reduction occurs. We can see from Table II that the most heavily weighted paths in (n,N) space will be the ones in which the largest values of N

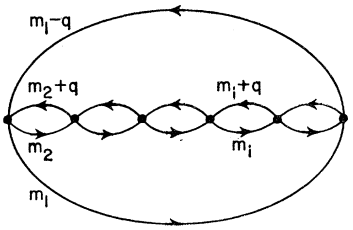

FIG. 4. A successive hole, filled-state interaction diagram.

are reached. If type V combinations are used, a value of $N=n$ can be reached and, hence, we get

$$\text{weight } (FFF\cdots EEE)\simeq 4^n\binom{n}{\frac{1}{2}n!}^4. \tag{3.4}$$

Similarly, using types II, III, or IV we get again (3.4). Except for III and IV the 4^n becomes $(4.5)^n$. The total number of allowed paths in (n,N) space is less than $\frac{1}{2}$ the number of random walks in one dimension which return to the origin in n steps, where steps of length 0, 1, and 2 are allowed. This is easily computed[11] as

$$\begin{aligned}\tfrac{1}{2}P(0,n)&=\frac{1}{4\pi}\int_0^{2\pi}[e^{-2ik}+e^{-ik}+1+e^{+ik}+e^{2ik}]^n dk\\&=\frac{1}{4\pi}\int_U^{2\pi}\left(\frac{\sin\frac{5}{2}k}{\sin\frac{1}{2}k}\right)^n dk,\end{aligned} \tag{3.5}$$

which, by the method of steepest descents[12] is, for large n, approximately $5^n(16\pi n)^{-1/2}$. Thus, the total number of diagrams can increase no faster than

$$(23)^n[(n/2)!]^4\approx(1.5)^n(2n!). \tag{3.6}$$

On the other hand, using only the most heavily weighted path with type II contributions only, we have at least as many as (3.4). Hence, the number of diagrams is asymptotically something like $AB^n(2n!)$. According to the data in Table I, $A\approx 0.12$, $B\approx 1.0$.

Having enumerated the diagrams, the next step is to bound each individual diagram. There are basically two possible types of divergences which the diagrams could possess. The first would be singularity for a finite value of the internal momenta, and the second a divergence for an infinite value of the internal momenta. We will examine now the first type of situation. For the many-body perturbation-theory diagrams, the action of the exclusion principle always prevents any divergence from a finite internal momentum plane. We will illustrate this situation for a case in which there is apparently an arbitrarily strong singularity. The diagram is given in Fig. 4. The contribution to E/N of the direct (no exchanges) term is

$$\begin{aligned}&\frac{(-1)^{n-1}}{(2\pi)^{3n}}\left(\frac{3}{4\pi k_F{}^3}\right)\int d^3q\prod_{i=1}^{n}(d^3m_i)\\&\qquad\times[v(q)]^n\Big/\prod_{i=2}^{n}\left[\mathbf{q}\cdot(\mathbf{q}+\mathbf{m}_i-\mathbf{m}_1)\frac{\hbar^2}{m}\right],\end{aligned} \tag{3.7}$$

$$|\mathbf{m}_1-\mathbf{q}|>k_F,\quad |\mathbf{m}_i+\mathbf{q}|>k_F,\quad 1<i\leq n,$$

$$|m_i|\leq k_F,\quad 1\leq i\leq n,$$

where we mention that we are treating the case without spin or isotopic spin, and k_F is the magnitude of the wave vector at the top of the Fermi sea. Apparently (3.7) diverges like q^{2-n} as $|\mathbf{q}|$ goes to zero. However, following Hugenholtz,[13] if we neglect quadratic terms in q we may convert (3.7) into

$$\begin{aligned}&3\left(-\frac{m}{\hbar^2}\right)^{n-1}(2\pi)^{-2n}k_F{}^{n-2}\int_0^\infty dq\,q^3[v(q)]^n\int_0^1\cdots\\&\qquad\int_0^1\mu_1 d\mu_1\prod_{i=2}^{n}\frac{\mu_i d\mu_i}{(\mu_i+\mu_1)},\end{aligned} \tag{3.8}$$

where the μ_i are cosines of the angles between $\mathbf{m}_i$ and $\mathbf{q}$ (except $\mu_1=-\cos$ine). We see from (3.8) that the integrand actually vanishes at $q=0$ instead of becoming singular. The problem of a vanishing denominator ($\mu_1=\mu_i=0$) does not cause any difficulty for, doing the integrals over μ_i, $i=2,\cdots,n$, we obtain

$$\begin{aligned}&3\left(-\frac{m}{\hbar^2}\right)^{n-1}(2\pi)^{-2n}k_F{}^{n-2}\int_0^\infty dq\,q^3[v(q)]^n\\&\qquad\times\int_0^1\mu_1 d\mu_1[1-\mu_1\ln(1+\mu_1)+\mu_1\ln\mu_1]^n.\end{aligned} \tag{3.9}$$

Since the nth root of the integrand for μ_1 is bounded, the entire term increases only geometrically as n goes to infinity.

As mentioned earlier, Fig. 3 may by itself diverge. We will now consider the bubble diagrams in more detail. First, the contribution of a bubble on a hole line has the opposite sign from a bubble on a filled-state line. This change of sign follows because they arise from the subtraction terms in the Rayleigh-Schrödinger perturbation theory.[14] That the bubble on hole lines cannot arise directly follows immediately from the second quantization formulation,[10] as it involves emptying a state twice. The effect of a bubble on a filled-state line of

[11] See, for instance, E. Montroll, *The Theory of Neutral and Ionized Gas*, edited by C. deWitt (John Wiley & Sons, Inc., New York, 1960), p. 110.

[12] See, for instance, H. Jeffreys and B. S. Jeffreys, *Methods of Mathematical Physics* (Cambridge University Press, New York, 1950), Chap. 17.

[13] N. M. Hugenholtz, Physica **23**, 533 (1957), especially Eq. (3.6).

[14] See, for example, K. A. Brueckner, *The Many Body Problem*, edited by C. de Witt (John Wiley & Sons, Inc., New York, 1959), p. 65 *et seq.*

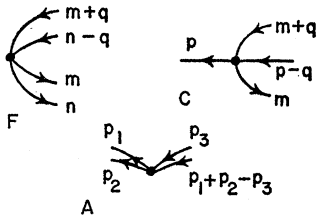

FIG. 5. Labeled "free" momentum-originating vertices.

momentum $\mathbf{k}$ is to insert a vertex contribution of

$$\frac{2}{(2\pi)^3}\int d^3m[v(0)-v(\mathbf{k}+\mathbf{m})], \quad |m|<k_F \tag{3.10}$$

and raise the power of the denominator by unity. On a hole line, there is the same contribution as (3.10) except that the over-all sign is now minus and $\mathbf{k}$ now stands for the momentum of the unoccupied state. Hence, summing over bubbles on all lines, we obtain for the contribution of a J vertex a factor in the integrand of

$$\frac{2}{(2\pi)^3}\int_{|m|<k_f} d^3m \times\sum_{i=1}^{j}[v(\mathbf{k}_i+\mathbf{m})-v(\mathbf{m}_i+\mathbf{m})]/\sum_{i=1}^{j}(\mathbf{k}_i^2-\mathbf{m}_i^2), \tag{3.11}$$

where j is the number of holes or filled-state lines present, $\mathbf{k}_i$ the filled-state momenta, and $\mathbf{m}_i$ the hole momenta. The $v(0)$ terms have cancelled as the number of holes must equal the number of filled-state lines in this type of diagram. Since v depends only on the magnitude of its argument, it follows easily that the integrals of v appearing in (3.11) depend only on $|\mathbf{k}_i|$ and $|\mathbf{m}_i|$. Furthermore, since $\mathbf{k}_i^2>k_F^2\geq\mathbf{m}_i^2$, the denominator may vanish only when all the momenta lie in the Fermi surface. But, for that case the numerator also clearly vanishes. Hence, the contribution of a summed bubble vertex is bounded by a quantity related to the first derivative of the integral of v appearing in (3.11). If the bubble diagrams are not summed, singularities may arise from, for instance, the $v(0)$ terms.[15]

We have seen, from analyzing the different types of apparent singularities that may occur for finite internal momenta, that they do not contribute any additional divergence to the perturbation series.

In order to study the possible divergence of a diagram at infinite values of the internal momenta, we will consider each type of vertex which is allowed by the Pauli exclusion principle to originate a momentum which may be infinite. These vertices are A, C, and F. First, if we label vertex F as shown in Fig. 5, then the vertex contributes $v(q)$ and the denominator following it has a non-negative part plus $2\mathbf{q}\cdot(\mathbf{q}+\mathbf{m}-\mathbf{n})$. As $v(q)$ goes to zero like q^{-2}, and we are doing only three-dimensional integrals, this remark suffices to show that the integral over the "free" momenta $\mathbf{q}$ originated at F vertices must converge at infinity at least like dq/q^2. If we label a C vertex as shown in Fig. 5, then the vertex contributes again a factor of $v(q)$. If we consider only the new line $m+q$ and hole m, then the next denominator is something non-negative plus $q^2+2\mathbf{m}\cdot\mathbf{q}$. Again the integral over $\mathbf{q}$ converges at infinity at least like dq/q^2. The analysis of A is slightly more complex. Pick any hole $\mathbf{m}$, then we may write

$$\mathbf{p}_1=\mathbf{m}+\mathbf{q}_1, \quad \mathbf{p}_3=\mathbf{m}+\mathbf{q}_3. \tag{3.12}$$

In this notation $\mathbf{q}_3$ will be the "free" momentum created at this vertex. The vertex will contribute a factor $v(\mathbf{q}_3-\mathbf{q}_1)$ and the next denominator to the right will be something non-negative plus $q_3^2+2\mathbf{q}_3\cdot\mathbf{m}$. As we have shown above, the exclusion principle keeps this quantity from vanishing in an unfortunate manner for $\mathbf{q}_3$ and $\mathbf{m}$ finite, so we may replace the q_3^2 in the volume element divided by it with a constant for the purpose of bounding the integral over $\mathbf{q}_3$. Our task is thus reduced to bounding

$$\max_{\mathbf{q}_1}\int_0^\infty dq_3 d\Omega_3|v(\mathbf{q}_3-\mathbf{q}_1)|, \tag{3.13}$$

where Ω_3 is the solid angle for $\mathbf{q}_3$. A bound for (3.13) follows easily when we note that

$$|v(x)|\leq A/(1+Bx^2) \tag{3.14}$$

for a square well.

It now follows easily that, since there are no divergences on any of the integrations and there are at most $n+1$ internal 3-dimensional momenta in a diagram of nth order, any nth-order Goldstone diagram contributes at most

$$M_1(M_2)^n, \tag{3.15}$$

where M_1 and M_2 are determinable constants. To show this we may break a general diagram of the form

$$\int v_1\frac{1}{D_1}v_2\cdots v_{n-1}\frac{1}{D_{n-1}}v_n d\tau \tag{3.16}$$

down into a number of factors depending on (essentially) single variables only, in the general manner indicated above. Then each single integration may be bounded. We get something like

$$\max[|v_n|]\prod_{i=1}^{n-1}\left[\int\left|v_i\frac{1}{D_i^*}\right|\right]\int\int d\mathbf{m}d\mathbf{n}, \tag{3.17}$$

where $\mathbf{m}$ and $\mathbf{n}$ are the two holes which occur in every diagram. Hence, we have shown so far that the perturbation series diverges at worst like

$$M_3(2n)!(M_4)^n. \tag{3.18}$$

[15] For a more general discussion of the cancellation of divergences for finite internal momenta, see V. V. Tolmachev, Dokl. Akad. Nauk S.S.S.R. **141**, 582 (1961) [translation: Soviet Phys.—Dokl. **6**, 976 (1962)].

We shall now show that (3.18) can be greatly improved upon. Although the argument is not rigorous, we believe that it makes the result quite plausible. It is based on the observation that when many excited states are filled, the denominators are, on the average, much larger than when very few excited states are present. If the range of the various internal momenta is determined by the convergence properties of the potentials or the exclusion principle alone, and not by the energy denominators which depend on N, then we may think of each hole, filled-state line pair as contributing an average excitation energy. When there are N of these pairs, the denominator will be, on the average, N times as large.

We must now determine when the ranges are so restricted. Clearly, the energy range of every hole momentum is completely restricted by the Pauli exclusion principle. It therefore remains to check the "free" momenta. We can do this by the familiar procedure of counting powers at infinity of each "free" momentum q. To facilitate this power counting, we may compute from (3.1)

$$\int_{-1}^{+1} v(|\mathbf{q}_1-\mathbf{q}_2|)dx_{12}=\frac{4\pi Va}{q_1q_2} \times[j_0(a(q_1+q_2))-j_0(a(q_1-q_2))], \quad (3.19)$$

where x_{12} is the cosine of the angle between $\mathbf{q}_1$ and $\mathbf{q}_2$. As the quantity in square brackets in (3.19) is bounded for all q_1 and q_2 by 2, we see that a vertex operator involving two "free" momenta may simply be divided equally between them, i.e., one inverse power for each one. Referring to the above analysis of the "free" momentum-creating vertices A, C, and F, we see (also let $\mathbf{p}_2=\mathbf{r}-\mathbf{q}_1$, with $\mathbf{r}$ possibly "free" for vertex A) that they contribute factors of A, q^{-1}; C, q^{-2}; and F, q^{-2}. A similar analysis of the vertices which may terminate a "free" momentum shows that they contributed factors of A, q^{-1}; B, q^{-1}; and E, q^{-2}, where q is the one "free" momentum which can be annihilated at the vertex. To be sure of not overlapping in our counting, we will consider only the originating and the terminating vertex for each "free" momentum. If we consider all 9 types of pairs of vertices, we find that all but AA and AB have at least 3 inverse powers of q, and hence determine the range of their respective "free" momentum. The other two pairs have only a factor of q^{-2} and so the range of q is possibly controlled by the energy denominators. By considering the integral[16]

$$\int_0^\infty \frac{dx}{(A+x^2)^n}=(\pi)^{1/2}\frac{\Gamma(n-\frac{1}{2})}{[2\Gamma(n)A^{n-\frac{1}{2}}]}, \quad (3.20)$$

we see that the approximate effect of an integration determined by the energy denominator is to multiply the magnitude of the result by $(A)^{1/2}$. When there are several denominators involved, the integration will effect the multiplication by $(\hat{A})^{1/2}$, where $\hat{A}$ is some kind of a mean A. If an A vertex occurs when there are N excited states, then we expect that the "free" momentum created will be annihilated on the average when there are of the order of N excited states present also. We also compute roughly the probability (fraction of diagrams) of the pairs AA, AB, and AE occurring. These are the only pairs which contain A as the first member. The number of such pairs is

$$AA=\binom{2}{2}=1,$$
$$AB=\binom{2}{2}\binom{N}{1}=N, \quad (3.21)$$
$$AE=\binom{2}{2}\binom{N}{2}=\tfrac{1}{2}N(N-1),$$

and their relative probability will be

$$AA=1/[\tfrac{1}{2}N(N+1)+1],$$
$$AB=N/[\tfrac{1}{2}N(N+1)+1], \quad (3.22)$$
$$AE=\tfrac{1}{2}N(N-1)/[\tfrac{1}{2}N(N+1)+1].$$

For vertex AE the contribution will be $1/N$ at each denominator times the number of diagrams. For AA and AB we must take only $1/(\hat{N})^{1/2}$ for first denominator to the right of first A vertex. Thus, averaging over the possible final vertices, we get a factor of

$$[\tfrac{1}{2}(N-1)+(N+1)/(\hat{N})^{1/2}]/[\tfrac{1}{2}N(N+1)+1] \quad (3.23)$$

for the first denominator to the right of A. This factor, however, tends to $1/N$ as N tends to infinity. The algebraic identity given by Hugenholtz[10]

$$\sum_{\text{permutation}} \xi_1^{-1}(\xi_1+\xi_2)^{-1}\cdots(\xi_1+\xi_2+\cdots+\xi_n)^{-1} = (\xi_1\xi_2\cdots\xi_n)^{-1} \quad (3.24)$$

is also suggestive in this regard as there are exactly $n!$ permutations. Hence, we may divide the weight factors given in (3.2) by N and multiply by some constant Λ in order to calculate the total contributions. When we calculate the contribution in this way, an analysis similar to that in (3.3) to (3.6) and Table II shows us that the nth order in perturbation theory diverges no faster than

$$\Gamma n!(\tilde{\Lambda})^n, \quad (3.25)$$

where $\tilde{\Lambda}$ is a multiple of Λ.

The so-called ladder insertions (several successive A vertices) form a special case. Integration over each "free" momentum (in the simple ladder diagram) in the region near the Fermi surface contributes a factor proportional to $\ln(\frac{1}{2}|\mathbf{m}+\mathbf{n}|+k_F-\frac{1}{2}|\mathbf{m}-\mathbf{n}|)$. For r

[16] W. Gröbner and N. Hofreiter, *Integraltafel, Zweiter Teil, Bestimmte Integrale* (Springer-Verlag, Berlin, 1958), No. 131.7.

successive such factors, the final integration over the hole momenta **m** and **n** gives a result proportional to $n!$. However, there are only 2^r of these terms, so they contribute terms at most of order $r!$. When a ladder insertion occurs with m hole lines present, its contribution is cut to the order of $(r!)/m^{r+1}(r\gg m)$ as can be seen from the integral $\int_0\cdots\int_0\prod^m dx_i \ln^r(\sum^m x_i)$. The rest of an nth-order diagram will contribute like a diagram of order $n-r$, or as we have seen $(n-r)!$. Hence, summing over all partitions we get $\sum r!(n-r)!$, which is again of order $n!$. Thus, although the ladder diagrams contribute more than their proportional amount, they will not increase the rate of divergence of the series as a whole. Summing out the ladder insertions with a K matrix will not render the whole series convergent, as can be seen by counting up the remaining contributions.

We believe that, although for certain potentials (such as S-state interaction only, which does not lead to a collapsed state and, hence, yields a convergent series) cancellation between terms may occur so as to decrease the rate of divergence, there is not a better general bound than (3.25). The derivation of Eq. (3.25) did not depend strongly on the properties of the potential, but only on $|v(q)|$ being bounded and going to zero like q^{-2} as q went to infinity. According to recent work of J. L. Gammel[17] the terms selected by the Bardeen-Cooper-Schrieffer theory of superconductivity are all present in the perturbation theory for an N-body system where the limit $N\to\infty$ is not taken. They are not of order 1, but higher order in $1/N$. The sum of these terms gives a contribution to the energy (for V small) of[18]

$$Ae^{-B/V} \tag{3.26}$$

in the limit as N goes to infinity. Since we have shown in the previous section that the radius of convergence tends to zero as N tends to infinity, probably like $N^{-\gamma}$ with $\frac{1}{3}\leq\gamma\leq 1$, let us guess that (3.26) is, for N finite, of a form like

$$A\exp\{-B/[V+CN^{-\gamma}]\}. \tag{3.27}$$

It now follows from examining the derivatives at $V=0$, that as N tends to infinity, the nth derivative is of the order of $(n)!$ for some value of N. Hence, these terms alone would, in general, prevent one from obtaining a better bound than (3.25).

4. SUMMABILITY OF THE PERTURBATION SERIES

Let us now consider the problem of assigning a meaning to the asymptotic perturbation-series expansion for the infinitely-many-body problem. According to Hardy,[19] Carleman has proved that a necessary and sufficient condition for

$$|g(z)|\leq\alpha_n{}^n|z|^n,\quad (|z|\leq r_0<\infty),\quad |\arg z|\leq\tfrac{1}{2}\pi \tag{4.1}$$

to imply $g(z)=0$, is that $\sum\alpha_n{}^{-1}$ diverge (for suitably regular α). If the many-body perturbation is asymptotic in the angular sector $|\arg V|\leq\frac{1}{2}\pi$, then this theorem means, since $(n!)^{1/n}\propto n$ implies $\sum\alpha_n{}^{-1}$ diverges, that there is at most one function which is regular for real, positive V and asymptotically equal to the perturbation series. That the desired solution, for a repulsive square-well potential, is regular follows directly from a perturbation-theory calculation of the first derivative for any real, positive V. We shall now investigate the determination of this function.

Let the energy per particle in the N-body problem with a square-well interaction be

$$E_N(V)=\sum_{n=0}^{\infty}{}_Ne_nV^n. \tag{4.2}$$

This series has a radius of convergence of the order of $N^{-\gamma}$, $\frac{1}{3}\leq\gamma\leq 1$, as was shown in Sec. 2, and, hence, by analytic continuation defines $E_N(V)$ everywhere except at singular points or on branch cuts. This analytic continuation is conveniently given by Mittag-Leffler's method.[19] Let

$$E_N(V,\delta)=\sum_{n=0}^{\infty}\frac{{}_Ne_nV^n}{\Gamma(\delta n+1)}. \tag{4.3}$$

For every $\delta>0$, $E_N(V,\delta)$ is an entire function of V as $E_N(V)$ is analytic at $V=0$. Also, as shown by Hardy,[19] Theorem 135,

$$\lim_{\delta\to 0}E_N(V,\delta)=E_N(V) \tag{4.4}$$

uniformly in any closed and bounded region in the Mittag-Leffler star of $E_N(V)$. The Mittag-Leffler star is defined by cutting the complex V plane from every singularity to infinity along rays. As we have pointed out above, $E_N(V)$ is regular for positive real V. Therefore, the positive real axis is interior to the Mittag-Leffler star for every N. The meaning which we wish to assign to the sum of the perturbation series for infinitely many bodies is

$$\lim_{N\to\infty}E_N(V)=\lim_{N\to\infty}\lim_{\delta\to 0}E_N(V,\delta). \tag{4.5}$$

We would like to interchange the order of limits in (4.5) to $\lim_{\delta\to 0}\lim_{N\to\infty}$. For this interchange to be correct, it is sufficient to show that $\lim_{N\to\infty}$ is uniform for δ small enough. To this end let us introduce

$$\phi_\delta(z)=\sum_{n=0}\frac{z^n}{\Gamma(1+\delta n)}\xrightarrow[|z|\to\infty]{}(1-z)^{-1},\qquad \tfrac{1}{2}\pi\delta\leq|\arg z|\leq\pi. \tag{4.6}$$

[17] J. L. Gammel (private communication). We wish to thank this author for making this information available to us prior to publication.

[18] See, for example, L. N. Cooper, *Lecture Notes on the Many-Body Problem from the First Bergen International School of Physics, 1961*, edited by C. Fronsdal (W. A. Benjamin, Inc., New York, 1962), p. 49.

[19] G. H. Hardy, *Divergent Series* (Oxford University Press, New York, 1956), Chap. VIII.

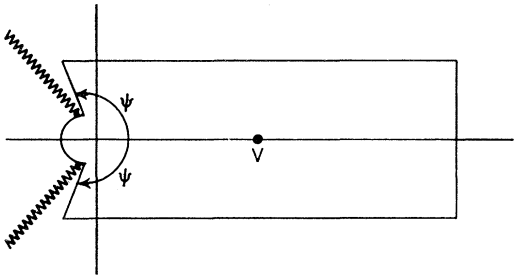

FIG. 6. N-dependent integration contour.

Now for V less than the radius of convergence we have by Cauchy's theorem for regular functions of a complex variable

$$E_N(V,\delta)=\frac{1}{2\pi i}\oint \phi_\delta\left(\frac{V}{u}\right)E_N(u)\frac{du}{u}, \tag{4.7}$$

where the contour is a circle about the origin outside V and inside the radius of convergence of E_N. Again, using Cauchy's theorem, we may deform the contour until it looks as in Fig. 6, and move V out along the positive, real axis. This procedure gives us the analytic continuation to any real and positive V. The angle ψ is picked so as to avoid all singularities of $E_N(u)$. Let us choose $\pi\delta<2\psi$. Then, for $\psi\leq|\arg(V/u)|\leq\pi$,

$$\frac{1}{u}\phi_\delta\left(\frac{V}{u}\right)\rightarrow\frac{1}{u-V} \tag{4.8}$$

as V/u tends to infinity. Hence, $\phi_\delta(V/u)$ is bounded on the contour of Fig. 6. We may now shrink the circular arc part of the contour to the origin. Since $E_N(u)$ is bounded, Eq. (4.8) holds and the length of the arc goes to zero, the contribution from this part of the contour goes to zero. Hence, the contour in Fig. 6 may be replaced by that in Fig. 7, which is independent of N. Now $E_N(u)$ tends to a finite limit at every point of this contour, and hence it does so uniformly. Therefore, we get that $E_N(V,\delta)$ tends uniformly for all $\pi\delta<2\psi$ to

$$E(V,\delta)=\frac{1}{2\pi i}\oint \phi_\delta\left(\frac{V}{u}\right)E(u)\frac{du}{u}, \tag{4.9}$$

where the contour is that of Fig. 7. By the properties of $\phi_\delta(x)$, $E(V,\delta)$ tends to $E(V)$ uniformly for V in any closed and bounded set interior to the contour. Thus,

$$E(V)=\lim_{\delta\rightarrow 0}E(V,\delta)=\lim_{\delta\rightarrow 0}\lim_{N\rightarrow\infty}E_N(V,\delta). \tag{4.10}$$

Also $E(V,\delta)$ is an analytic function of δ for $0<\delta<2\psi/\pi$.

There is a uniform bound in N of the type (3.25) for the N-body perturbation series, in addition to the nth coefficient being bounded by

$$A(BN^\gamma)^n. \tag{4.11}$$

This result can be seen by remembering that the perturbation expansion for the N-body problem in a box can be obtained from that for the infinitely-many-body problem by replacing the integrals with sums with certain terms deleted. The replacement of integrals by sums does not change the order of magnitude of a convergent integral. The deletions have little effect until n (the order) is of the order of magnitude of N, where their effect is to reduce the rate of increase from that given by Eq. (3.25) to that given by Eq. (4.11).

Because of the uniform bound like (3.25), we see that the series for $E_N(V,\delta)$ converges absolutely and uniformly for all V if $\delta>1$. If ψ is greater than $\frac{1}{2}\pi$, then we have $E_N(V,\delta)$ an analytic function of δ for all $\delta>0$, and we can analytically continue it from $\delta>1$ where we may calculate directly from its series expansion to $\delta=0$ where that expansion diverges.

As we saw in Sec. 2, if the density is sufficiently low (small compared to the jamming density for hard spheres), we may reasonably expect to exclude all singular points from the right-half, complex V plane, including the imaginary axis. If this situation prevails, we may introduce

$$\begin{aligned}h_N(t,\delta)&=\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}e^{wt}E_N(w^{-1},\delta)\frac{dw}{w}\\&=\sum_{n=0}^{\infty}\frac{Ne_nt^n}{n!\,\Gamma(1+\delta n)}.\end{aligned} \tag{4.12}$$

Since $E_N(V,\delta)$ approximately equals[19] $E_N(V)$ as $V\rightarrow\infty$ in $|\arg z|<\psi(\rho)-\frac{1}{2}\pi\delta$, where $\psi(\rho)$ is now the minimum angle made by a line joining the origin and a singularity of $E_N(V)$, and the positive real axis, and since from Sec. 2 we expect, for low enough density, the branch of $E_N(V)$ obtained by continuation for $\mathrm{Re}(V)>0$ to tend to a finite limit, we see that the integral definition (4.12) implies $h_N(t,\delta)$ is bounded by $Me^{\epsilon t}$ for any $\epsilon>0$. We take $\epsilon>0$ to miss the pole at $w=0$. As long as $\psi(\rho)>\pi/2$, the $E_N(w^{-1},\delta)$, as we have seen above, tend uniformly in N as $N\rightarrow\infty$ to $E(w^{-1},\delta)$ on the imaginary axis, and since (4.12) converges absolutely [$E(w^{-1},\delta)\propto w^{-1}$ as $w\rightarrow\infty$], $h_N(t,\delta)$ also tends uniformly to a limit $h(t,\delta)$ as $N\rightarrow\infty$. From the series expansion (4.12) we see that $h(t,\delta)$ is entire. Hence we can, from

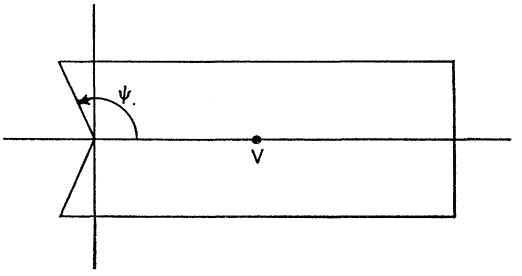

FIG. 7. N-independent integration contour.

the series expansion, directly compute by the Laplace transform theorems,[20]

$$\lim_{N\to\infty} E_N(V,\delta)=\lim_{N\to\infty}\int_0^\infty e^{-t}h_N(Vt,\delta)dt,$$
$$E(V,\delta)=\int_0^\infty e^{-t}h(Vt,\delta)dt. \tag{4.13}$$

Our bound for $h(Vt,\delta)$ insures that (4.13) converges absolutely for any finite V and uniformly in δ, as $\delta\to 0$. We may, therefore, interchange integration and the limit $\delta\to 0$. We recognize that taking the limit $\delta\to 0$ applied to $h(Vt,\delta)$ merely effects the analytic continuation to all points in the Mittag-Leffler star. We will therefore take the limit as $\delta\to 0$ and mean by $h(x)$ the analytic continuation of its power-series definition. Hence, if $\psi(\rho)>\frac{1}{2}\pi$, then

$$h(x)=\sum_{n=0}^{\infty} e_n x^n/(n!),$$
$$E(V)=\int_0^\infty e^{-t}h(tV)dt, \tag{4.14}$$

which is the Borel sum[19] of the energy series. That $\psi(\rho)>\frac{1}{2}\pi$ is necessary follows by letting $\tau=tV$ in (4.14). This change of variables shows that $E(V)$ may be continued (since V may be arbitrarily large when real) throughout the whole region $\mathrm{Re}(V)>0$. $E(V)$ for V purely imaginary can be defined by taking the limit as $\mathrm{Re}(V)\to 0^+$.

It should be remarked that, for more general potentials, Carleman's theorem, Eq. (4.1), tells us that the condition for determining a unique sum is that the closed right half-plane be free of singularities, *in the neighborhood of the origin* rather than the whole right half-plane. We obtained the whole right half-plane for the repulsive square-well potential because of the behavior of $E(V)$ in the neighborhood of $V=+\infty$. In the case of more general potentials, Eq. (4.14) will sum $E(V)$ only in the so-called "Borel polygon of summability."[19] This polygon will not include all real, positive V if there are singularities in the open right half-plane. However, as long as it sums the series in the neighborhood of at least one point, it is sufficient, by analytic continuation, to determine, at least in principle, the function everywhere except at singularities, on branch cuts, or beyond natural barriers. It should also be noted that even for the square-well potential, Eq. (4.14) without the variable change may not converge in the whole right half-plane, but will for all real and positive V.

[20] See, for example, B. Van der Pol and H. Bremmer, *Operational Calculus Based on the Two-Sided Laplace Integral* (Cambridge University Press, New York, 1959).

In the limit of vanishing ρ we can work out explicitly the various functions in (4.14). According to Huang and Yang[21] the energy per particle to first order in ρ is

$$\frac{EMa^2}{\hbar^2}=\left\{1-\frac{\tanh[(VMa^2/\hbar^2)^{1/2}]}{(VMa^2/\hbar^2)^{1/2}}\right\}\frac{(k_F a)^3}{(3\pi)}. \tag{4.15}$$

Solving (4.12) for h, we obtain

$$h(x)=\frac{(k_F a)^3}{3\pi}\left[1-\frac{8}{\pi^2}\sum_{n=0}^{\infty}(2n+1)^{-2}\times\exp\{-4Ma^2x/[\hbar^2(2n+1)^2\pi^2]\}\right], \tag{4.16}$$

which goes to a constant (without oscillation) as $x\to\infty$. In the limit as $\rho\to 0$ we have a "large" convergent part, Eq. (4.15), plus a "small" asymptotic part. Hence, one would expect that approximations based on summing a subsequence of diagrams (for instance, Brueckner's[2,14]) would form reasonable approximations for very low density, but will very likely begin to fail as the density increases. Also, such an approximation may be valid for larger density and weak potentials. Although summing a subsequence of diagrams and expanding in terms of, say, Brueckner's[2,14] K matrix does not convert a divergent series into a convergent one, if the density is very low or K is very weak, a serviceable approximation may perhaps be obtained by taking only low order in K.

We see that (4.14) correctly gives the energy per particle for the infinitely-many-body problem so long as the closed right half-plane (except the origin) is free of singularities. As singularities move into the right half-plane, we no longer obtain the correct energy. What happens is that in going along the imaginary axis in (4.12) we get on the wrong branch of E_N. The energy obtained in (4.14) will, however, by the variational principle, always be an upper bound to the true energy, since the corresponding wave function can be obtained by performing a Borel summation on the wave matrix.[14] This result can be shown by use of Carleman's theorem [Eq. (4.1)] and a modified version of Goldstone's[3] proof that the perturbation expansion satisfies the Schrödinger equation.

An example of what can happen (for a different potential) when the condition $\psi(\rho)>\frac{1}{2}\pi$ is violated is afforded by the BCS theory of superconductivity.[18] Even though, as Katz[7] has shown, the singularities of the N-body problem are branch points at which the energy is continuous, in the limit as $N\to\infty$ they can conspire to give an essential singularity of the form[18]

$$E_0(V)=Ae^{-B/V}. \tag{4.17}$$

The corresponding $h(x)$ is

$$h(x)=0, \quad x<B$$
$$=A, \quad x\geq B, \tag{4.18}$$

[21] K. Huang and C. N. Yang, Phys. Rev. **105**, 767 (1957).

which is *not* an analytic function of x. It is perhaps illuminating to consider how this circumstance could have arisen, since (4.12) and (4.13) with subscripts N are exact. Let us consider

$$f_N(v,\delta)=\sum_{n=0}^{\infty}\frac{{}_Nf_nV^n}{\Gamma(1+\delta n)}, \tag{4.19}$$

where the ${}_Nf_n$ are the series-expansion coefficients of (3.27). When $\delta\ll 1$, $f_N(V,\delta)$ is a close approximation to (3.27). When $\delta=1$, $h(x)$ given in (4.18) is the limit as $N\to\infty$ of $f_N(V,1)$. This function still bears a slight resemblance to (3.27). When $\delta>1$, one can easily show that $f_N(V,\delta)$ tends to a limit for every value of V which can be as closely approximated by the series (4.19) with f_n replacing ${}_Nf_n$ as one pleases. Since all $f_n=0$, this means $f_N(V,\delta)\to 0$ for all V if $\delta>1$. Yet

$$\lim_{N\to\infty}\lim_{\delta'\to 0}\int_0^{\infty}e^{-t}\left\{\sum_{n=0}^{\infty}\frac{{}_Nf_nV^nt^{n\delta}}{\Gamma(1+n\delta)\Gamma(1+n\delta')}\right\}dt$$

$$=\lim_{N\to\infty}A\exp[-B/(V+CN^{-\gamma})]=Ae^{-B/V}. \tag{4.20}$$

What happens is that $f_N(V,\delta)$ has a large peak at a distance from the origin of order N^{γ} so, although $f_N(V,\delta)\to 0$, the integral does not. Hence, when the series is not asymptotic in $|\arg z|\leq\frac{1}{2}\pi$, the interchange of the integration and the limit as N goes to infinity in (4.13) is not valid.

Physically, where do these effects arise? Since, for low enough density, we solve correctly the repulsive square-well problem, they do not always arise as a result of the solution procedures alone. The effects come from situations where the effect on every power of V tends to zero as N tends to infinity, but not uniformly so. Consequently, they do not involve any finite number of excited states. Significant effects occur when the number of excited states is of the order of $N^{\gamma'}$. This sort of phenomena is usually related to the onset of long-range order in the physical system. We feel that the perturbation procedures described herein are applicable to many-body systems, so long as there is no condensation, collapsed state ($V<0$), superconductivity, superfluidity, phase transition, etc. For the infinitely repulsive, square-well potential problem there is probably an order-disorder transition[22] to a crystalline structure at a density somewhat below the jamming density for randomly arranged spheres.[23] This transition probably sets the limit of validity for the procedure given in (4.14), although one may continue to calculate a metastable state beyond that point.

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[22] B. J. Alder and T. E. Wainwright, Phys. Rev. **127**, 359 (1962).
[23] K. O. Rice, J. Chem. Phys. **12**, 1 (1944).